

# Reconstruction of Support of a Measure From Its Moments

A. M. Jasour and C. Lagoa

## Abstract

In this paper, we address the problem of reconstruction of support of a measure from its moments. More precisely, given a finite subset of the moments of a measure, we develop a semidefinite program for approximating the support of measure using level sets of polynomials. To solve this problem, a sequence of convex relaxations is provided, whose optimal solution is shown to converge to the support of measure of interest. Moreover, the provided approach is modified to improve the results for uniform measures. Numerical examples are presented to illustrate the performance of the proposed approach.

## I. INTRODUCTION

In this paper, we aim at solving the problem of reconstructing of support of a measure using only its moments. More precisely, we consider the following problem.

*Problem 1:* Given the moment sequence of a measure  $\mu$ , find a polynomial  $\mathcal{P} : \mathbb{R}^n \rightarrow \mathbb{R}$  such that the set

$$\mathcal{K} = \{x \in \mathbb{R}^n : \mathcal{P}(x) \geq 1\}$$

coincides with the support set of the measure  $\mu$ .

This problem has many applications in many different areas. A few examples are, problem of shape reconstruction from indirect measurements ([2], [17]), signal reconstruction from sparse measurements ([1], [4]), and problems in statistics [5]. Moreover, this problem can be applied in area of optimization. For example, moment approaches to polynomial optimization over semialgebraic sets where one aims at solving

$$\mathbf{f}^* = \sup_{x \in \mathbb{K}} f(x) \quad (1)$$

by looking at the moments of the measures in the following problem

$$\rho^* = \sup_{\mu_x \in \mathcal{M}(\mathbb{K})} \int_{\mathbb{K}} f(x) d\mu_x \quad (2)$$

$$\text{s.t.} \quad \int_{\mathbb{K}} d\mu_x = 1 \quad (2a)$$

requires one to extract an optimal solution by finding an  $x^*$  in the support set of the optimal solution  $\mu_x^*$  of the problem above; see [14], [15]. The same problem appears in semi-algebraic chance optimization problems of the form

$$\mathbf{P}^* = \sup_{x \in \chi} \text{Prob}_{\mu_q} \{q \in \mathbb{R}^m : f(x, q) \geq \gamma\} \quad (3)$$

which can also be solved using a moment approach and also require finding a point in the support set of a measure of which one only knows a finite set of moments; see [9], [10], [11].

In this paper, to reconstruct the support of the measure of interest from its moments, we develop a sequence of semidefinite programming (SDP) problems whose solutions converge to the solution of Problem 1.

Several approaches have been proposed to construct the support from the moments information. In [6] an approach to exact reconstruction of convex polytope supports is proposed, which is based on the collection of moment formulas combined with Vandermonde factorization of finite rank Hankel matrices. In [7], a method to reconstruct planar semi-analytic domains from their moments is proposed based on the diagonal Pade approximation where it can approximate arbitrarily closely any bounded domain. [13] provides an method to obtain a polynomial that vanishes on the boundary of support.

In this paper, we take a different approach. The proposed method relies on results on Sum of Squares (SOS) polynomials and also, results on necessary and sufficient condition for moment sequence to have a representing measure. A hierarchy of semidefinite relaxations for approximation of the support set is proposed.

The outline of the paper is as follows. In Section II, the notation used in this paper as well as preliminary results on measures theory and SOS polynomials are presented. In Section III, a convex formulation of support reconstruction problem as well as numerical examples is provided. In Section IV a modified SDP is given to improve the results for uniform measures. Concluding remarks are provided in Section V.

## II. NOTATION AND PRELIMINARY RESULTS

### A. Notations and Definitions

Let  $\mathbb{R}[x]$  be the ring of real polynomials in the variables  $x \in \mathbb{R}^n$ . Given  $\mathcal{P} \in \mathbb{R}[x]$ , we will represent  $\mathcal{P}$  as  $\sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha$  using the standard basis  $\{x^\alpha\}_{\alpha \in \mathbb{N}^n}$  of  $\mathbb{R}[x]$ , and  $\mathbf{p} = \{p_\alpha\}_{\alpha \in \mathbb{N}^n}$  denotes the sequence of polynomial coefficients. Moreover, let  $\Sigma^2[x] \subset \mathbb{R}[x]$  be the set of sum of squares (SOS) polynomials.  $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}$  is a SOS polynomial if it can be written as a sum of *finitely* many squared polynomials, i.e.  $\sigma(x) = \sum_{j=1}^{\ell} h_j(x)^2$  for some  $\ell < \infty$  and  $h_j \in \mathbb{R}[x]$  for  $1 \leq j \leq \ell$ . Given  $n$  and  $r$  in  $\mathbb{N}$ , we define  $S_{n,r} := \binom{r+n}{n}$  and  $\mathbb{N}_r^n = \{\alpha \in \mathbb{N}^n : \|\alpha\|_1 \leq r\}$ . Let  $\mathbb{R}_d[x] \subset \mathbb{R}[x]$  denote the set of polynomials of degree at most  $d \in \mathbb{N}$ , which is indeed a vector space of dimension  $S_{n,d}$ .

Let  $\mathbb{R}^{\mathbb{N}}$  denote the space of real sequences, and let  $\mathcal{M}(\mathcal{K})$  be the set of finite Borel measures  $\mu$  such that  $\text{supp}(\mu) \subset \mathcal{K}$ , where  $\text{supp}(\mu)$  denotes the support of the measure  $\mu$ ; i.e., the smallest set that contains all measurable sets with strictly positive  $\mu$  measure. A sequence  $\mathbf{y} = \{y_\alpha\}_{\alpha \in \mathbb{N}^n} \in \mathbb{R}^{\mathbb{N}}$  is said to have a *representing measure*, if there exists a finite Borel measure  $\mu$  on  $\mathbb{R}^n$  such that  $y_\alpha = \int x^\alpha d\mu$  for every  $\alpha \in \mathbb{N}^n$  -see [14], [15]. In this case,  $\mathbf{y}$  is called the moment sequence of the measure  $\mu$ . Given a square symmetric matrices  $A$ , the notation  $A \succcurlyeq 0$  denotes positive semidefiniteness of  $A$ .

**Putinars property:** A closed semialgebraic set  $\mathcal{K} = \{x \in \mathbb{R}^n : \mathcal{P}_j(x) \geq 0, j = 1, 2, \dots, \ell\}$  defined by polynomials  $\mathcal{P}_j \in \mathbb{R}[x]$  satisfies *Putinar's property* if there exists  $\mathcal{U} \in \mathbb{R}[x]$  such that  $\{x : \mathcal{U}(x) \geq 0\}$  is compact and  $\mathcal{U} = \sigma_0 + \sum_{j=1}^m \sigma_j \mathcal{P}_j$  for some SOS polynomials  $\sigma_j \in \Sigma^2[x]$  - see [12], [15]. Putinar's property holds if the level set  $\{x : \mathcal{P}_j(x) \geq 0\}$  is compact for some  $j$ , or if all  $\mathcal{P}_j$  are affine and  $\mathcal{K}$  is compact - see [12]. Clearly these results imply that if there exists  $M > 0$  such that the polynomial  $\mathcal{P}_{\ell+1}(x) := M - \|x\|^2 \geq 0$  for all  $x \in \mathcal{K}$ , then  $\mathcal{K} \cap \{x : \mathcal{P}_{\ell+1} \geq 0\}$  satisfies Putinar's property.

**Moment matrix:** Given  $r \geq 1$  and the sequence  $\{y_\alpha\}_{\alpha \in \mathbb{N}^n}$ , the moment matrix  $M_r(\mathbf{y}) \in \mathbb{R}^{S_{n,r} \times S_{n,r}}$ , containing all the moments up to order  $2r$ , is a symmetric matrix defined as follows [14], [15]:

$$M_r(\mathbf{y})(i, j) = y_{\alpha^{(i)} + \alpha^{(j)}}, \quad 1 \leq i, j \leq S_{n,r}, \quad (4)$$

where the elements of the moment sequence  $\mathbf{y} = \{y_\alpha\}_{\alpha \in \mathbb{N}^n}$  are sorted according to a graded reverse lexicographic order of the corresponding monomials so that we have  $\mathbb{R}^n \ni \mathbf{0} = \alpha^{(1)} < \dots < \alpha^{(S_{n,2r})}$  and  $S_{n,2r}$  is the number of moments in  $\mathbb{R}^n$  up to order  $2r$ .

For  $r = 2$  and  $n = 2$ , the moment matrix containing moments up to order  $2r$  is

$$M_2(\mathbf{y}) = \begin{bmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ - & - & - & - & - & - \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ - & - & - & - & - & - \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix} \quad (5)$$

**Localizing matrix:** Given a polynomial  $\mathcal{P} \in \mathbb{R}[x]$  with coefficient vector  $\mathbf{p} = \{p_\gamma\}_{\gamma \in \mathbb{N}^n}$  and degree  $\delta$ , localizing matrix  $M_r(\mathbf{p}\mathbf{y})$  with respect to  $\mathbf{y}$  and  $\mathbf{p}$  is as follows [14], [15]:

$$M_r(\mathcal{P}(x)\mathbf{y})(i, j) = \sum_{\gamma \in \mathbb{N}^n} p_\gamma y_{\gamma + \alpha^{(i)} + \alpha^{(j)}}, \quad 1 \leq i, j \leq S_{n,r}. \quad (6)$$

For example, given  $\mathbf{y} = \{y_\alpha\}_{\alpha \in \mathbb{N}^2}$  and polynomial  $\mathcal{P}$ ,

$$\mathcal{P}(x) = ax_1 - bx_2^2, \quad (7)$$

the localizing matrix for  $r = 1$  is formed as follows

$$M_1(\mathcal{P}(x)\mathbf{y}) = \begin{bmatrix} ay_{10} - by_{02} & ay_{20} - by_{12} & ay_{11} - by_{03} \\ ay_{20} - by_{12} & ay_{30} - by_{22} & ay_{21} - by_{13} \\ ay_{11} - by_{03} & ay_{21} - by_{13} & ay_{12} - by_{04} \end{bmatrix} \quad (8)$$

## B. Preliminary Results

In this section, we state some standard results found in the literature that will be referred to later. The following results give necessary, sufficient conditions for  $\mathbf{y}$  to have a representing measure  $\mu$  – for details see [8], [14], [15].

Consider the semialgebraic set  $\mathcal{K}$  defined as

$$\mathcal{K} = \{x \in \mathbb{R}^n : g_j(x) \geq 0, \quad j = 1, 2, \dots, \ell\}. \quad (9)$$

for some polynomials  $\mathcal{P}_j \in \mathbb{R}[x]$ , and assume that  $\mathcal{K}$  satisfies Putinar's property.

(i) If  $f \in \mathbb{R}[x]$  is strictly positive on  $\mathcal{K}$ , then:

$$f = \sigma_0 + \sum_{j=1}^l \sigma_j g_j \quad (10)$$

for some  $\sigma_j \in \Sigma^2[x]$ .

(ii) The sequence  $\mathbf{y} = \{y_\alpha\}_{\alpha \in \mathbb{N}^n}$  has a representing finite Borel measure  $\mu$  on  $\mathcal{K}$  if and only if:

$$M_r(\mathbf{y}) \succcurlyeq 0, M_r(g_j \mathbf{y}) \succcurlyeq 0, \quad j = 1, \dots, m \quad (11)$$

for every  $r \in \mathbb{N}^n$ .

### III. CONVEX FORMULATION

The approach presented in this paper relies on finding polynomial approximations of the indicator function of the support set of the measure of interest. More precisely, let  $\mathcal{K}$  represent the support set of a given measure  $\mu$ . The results in this paper aim at finding polynomial approximations of

$$\mathbb{I}_{\mathcal{K}}(x) \doteq \begin{cases} 1 & \text{if } x \in \mathcal{K} \\ 0 & \text{otherwise.} \end{cases}$$

and use the level sets of these polynomials to approximate  $\mathcal{K}$ . In order to approximate the indicator function above consider the following optimization problem.

*Problem 2:* Let  $d$  be a given integer. Moreover, let  $\mathcal{B}$  be a known (simple) set containing the support set  $\mathcal{K}$  and  $\mu_{\mathcal{B}}$  be the Lebesgue measure supported on the set  $\mathcal{B}$ . Solve

$$\mathbf{P}_2^* := \min_{\mathcal{P}_d(x) \in \mathbb{R}_d[x]} \int \mathcal{P}_d(x) d\mu_{\mathcal{B}} \quad (12)$$

$$\text{s.t. } \mathcal{P}_d(x) \geq 0, x \in \mathcal{B} \quad (12a)$$

$$\mathcal{P}_d(x) \geq 1, x \in \mathcal{K}. \quad (12b)$$

For every  $d$ , the problem above provides a polynomial  $\mathcal{P}_d^*$  with the smallest  $\ell_1$ -norm on  $\mathcal{B}$  that is i) positive in the (simple) set  $\mathcal{B}$  and ii) larger than one inside the support set  $\mathcal{K}$ . For this (infinite dimensional) optimization problem we have the following result.

*Theorem 1:* For a given integer  $d$ , let

$$\mathcal{K}_d \doteq \{x \in \mathbb{R}^n : \mathcal{P}_d^*(x) \geq 1\}$$

be the semialgebraic set constructed using the solution  $\mathcal{P}_d^*$  of the problem (12). Then

$$\lim_{d \rightarrow \infty} \mu_{\mathcal{B}}(\mathcal{K}_d - \mathcal{K}) = 0.$$

*Proof:* As in [3] one can show that  $\mathcal{P}_d^*$  converges almost uniformly (with respect to measure  $\mu_{\mathcal{B}}$ ) to the indicator function  $\mathbb{I}_{\mathcal{K}}$ . Moreover, one has  $\mathcal{K} \subseteq \mathcal{K}_d$  for all  $d$ . These two facts imply that

$$\lim_{d \rightarrow \infty} \mu_{\mathcal{B}}(\mathcal{K}_d - \mathcal{K}) = 0$$

which completes the proof. ■

In the optimization problem above, one approximates the indicator function of the set  $\mathcal{K}$  by using the knowledge that this set is contained in a known set  $\mathcal{B}$ . This set is usually chosen in such a way that one can compute all the moments of the measure  $\mu_{\mathcal{B}}$  in a closed form.

However, the problem above obviously requires the knowledge of the measure  $\mu$  whose support  $\mathcal{K}$  we are trying to determine. To be able to solve this problem by using only knowledge of moments consider a bounding set  $\mathcal{B}$  defined by a set of polynomial inequalities; i.e.,

$$\mathcal{B} = \{x \in \mathbb{R}^n : g_j(x) \geq 0, j = 1, \dots, l\}$$

where  $g_j, j = 1, 2, \dots, l$  are given polynomials. As before, let  $\mu_{\mathcal{B}}$  be the Lebesgue measure supported in  $\mathcal{B}$  with  $\alpha$ -th moment  $y_{\mathcal{B}_\alpha}$ . Moreover, let the (infinite) vector  $\mathbf{y}$  be the vector containing all the moments of the measure  $\mu$ . Then, define the following optimization problem (which has an infinite number of constraints).

*Problem 3:*

$$\mathbf{P}_3^* := \min_{p_\alpha, \sigma_j} \sum_{\alpha=0}^d p_\alpha y_{\mathcal{B}_\alpha} \quad (13)$$

$$\text{s.t. } \mathcal{P}_d(x) = \sum_{\|\alpha\|_1 \leq d} p_\alpha x^\alpha \quad (13a)$$

$$\mathcal{P}_d(x) = \sigma_0(x) + \sum_{j=1}^l \sigma_j(x) g_j(x) \quad (13b)$$

$$\sigma_j \in \Sigma^2[x]; j = 0, 1, \dots, l \quad (13c)$$

$$\deg(\sigma_0) \leq d; \quad (13d)$$

$$\deg(\sigma_j g_j) \leq d; j = 1, 2, \dots, l \quad (13e)$$

$$M_\infty((\mathcal{P}_d(x) - 1)\mathbf{y}) \succcurlyeq 0 \quad (13f)$$

The problem above is a first step towards an implementable version of Problem 2. The objective function is the same in both, just represented as a function of the moments of  $\mu_{\mathcal{B}}$  in Problem 3. Constraint (13b) enforces  $\mathcal{P}_d$  to be positive on the set  $\mathcal{B}$ . Finally, given the definition of localization matrix, constraint (13f) ensures that  $\mathcal{P}_d$  is larger than one in the support set of  $\mu$ .

Since one cannot solve the problem above, in this paper we propose the following relaxation.

*Problem 4:*

$$\mathbf{P}_4^* := \min_{p_\alpha, \sigma_j} \sum_{\alpha=0}^d p_\alpha y_{\mathcal{B}_\alpha} \quad (14)$$

$$\text{s.t. } \mathcal{P}_d(x) = \sum_{\|\alpha\|_1 \leq d} p_\alpha x^\alpha \quad (14a)$$

$$\mathcal{P}_d(x) = \sigma_0(x) + \sum_{j=1}^l \sigma_j(x) g_j(x) \quad (14b)$$

$$\sigma_j \in \Sigma^2[x]; j = 0, 1, \dots, l \quad (14c)$$

$$\deg(\sigma_0) \leq d; \quad (14d)$$

$$\deg(\sigma_j g_j) \leq d; j = 1, 2, \dots, l \quad (14e)$$

$$M_r((\mathcal{P}_d(x) - 1)\mathbf{y}) \succcurlyeq 0 \quad (14f)$$

where,  $r \geq 1$  is relaxation order. In other words, we truncate the infinite moment localization matrix. One should note that the problem above can be formulated as a standard SDP; i.e., minimization of a linear function subject to Linear Matrix Inequalities (LMIs).

The truncation of the moment localization matrix provides an approximation of the constraint  $\mathcal{P}_d(x) \geq 1$  for all  $x \in \mathcal{K}$ . Although, if  $r$  is “large” one has acceptable estimates of the support set, for “low” values of  $r$  this can lead to estimates of the support set that are less accurate than desirable.

**Example 1:** Let,  $\mathbf{y}$  be a moment sequence of uniform probability measure  $\mu$  supported on  $[-0.5, 0.5]$ . The  $\alpha$ -th moment of uniform distribution  $U[a, b]$  is  $y_\alpha = \frac{b^{\alpha+1} - a^{\alpha+1}}{(\alpha+1)(b-a)}$ . For this example, we take  $\mathcal{B} = [-1, 1]$ , and use the moments up to order  $2d$ . To solve the SDP (14), Yalmip is used which is a Matlab-based toolbox aimed at optimization [16]. The obtained results are depicted in Fig 1. One can see as  $d$ , the order of polynomial, increases  $\mathcal{P}_d(x)$  converges to indicator function of support of uniform measure. Hence, the semialgebraic set  $\mathcal{K}_d = \{x \in \mathbb{R} : \mathcal{P}_d(x) \geq 1\}$  provides better approximations of the support as one increases  $d$ . However, as one can see in Fig 1,  $\mathcal{P}_d$  can be below one in a significant subset of the support of  $\mu$ .

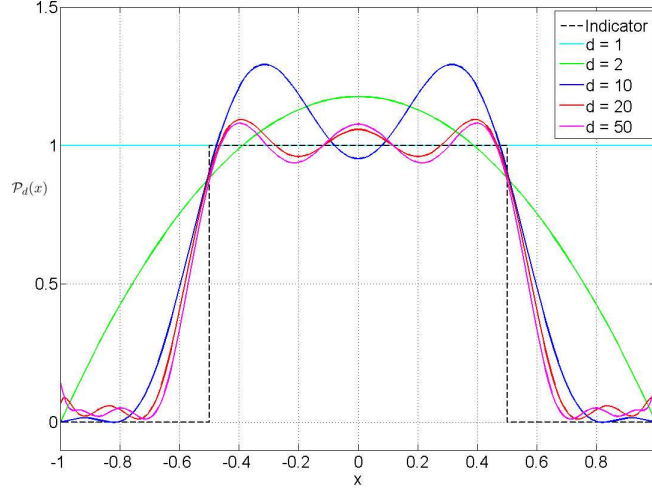


Fig. 1. Result of SDP in (14) For Example 1

#### A. An Heuristic for Improved Performance

To minimize the measure of the subset of the support of the measure  $\mu$  where  $\mathcal{P}_d$  is below one, we propose to maximize the values of  $\mathcal{P}_d(x)$  inside the support of the measure while still trying to bring its values as low as possible everywhere else in  $\mathcal{B}$ . This results in following modified SDP.

*Problem 5:*

$$\mathbf{P}_5^* := \min_{p_\alpha, \sigma_j} \sum_{\alpha=0}^d p_\alpha y_{\mathcal{B}_\alpha} - \omega_h h \quad (15)$$

$$\text{s.t. } \mathcal{P}_d(x) = \sum_{\|\alpha\|_1 \leq d} p_\alpha x^\alpha \quad (15a)$$

$$\mathcal{P}_d(x) = \sigma_0(x) + \sum_{j=1}^l \sigma_j(x) g_j(x) \quad (15b)$$

$$\sigma_j \in \Sigma^2[x]; j = 0, 1, \dots, l \quad (15c)$$

$$\deg(\sigma_0) \leq d; \quad (15d)$$

$$\deg(\sigma_j g_j) \leq d; j = 1, 2, \dots, l \quad (15e)$$

$$M_r((\mathcal{P}_d(x) - h)\mathbf{y}) \succcurlyeq 0 \quad (15f)$$

$$1 \leq h \leq 1 + \Delta h \quad (15g)$$

where,  $\omega_h$  and  $\Delta h$  are positive design parameters.

To show the effectiveness of the modified SDP, we again consider the uniform measure in Example 1. Fig 2 shows the results obtained by solving the modified SDP with parameters  $\omega_h = 1.2$  and  $\Delta h = 0.2$ . As it is seen, one obtains a substantial improvement in the estimate of the support set.

**Example 2:** In this example, we consider a  $\text{Beta}(4, 4)$  probability measure on  $[0, 1]$ . The  $\alpha$ -th moment of Beta distribution  $\text{Beta}(a, b)$  over  $[0, 1]$  is  $y_\alpha = \frac{a+b-1}{(a+b+\alpha-1)} y_{\alpha-1}$  and  $y_0 = 1$ . We assume that set  $\mathcal{B} = [-1.2, 1.2]$ , and use the moments up to order  $2d$ . The obtained results by solving SDP (15) with parameters  $\omega_h = 1.2$  and  $\Delta h = 0.2$  are depicted in Fig 3.

This is a more difficult problem than previous ones since, in terms of probability, there is a “smooth transition” from the interior to the exterior of the support set. Nevertheless, if one uses enough

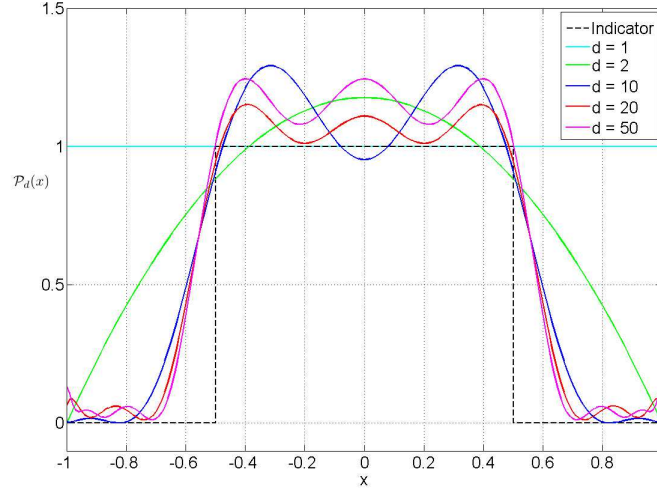


Fig. 2. Result of SDP in (15) For Example 1

moments, one can get a very good approximation of the support.

**Example 3:** In here, we consider a 2-dimensional example where one wants to approximate the support of a uniform probability measure on  $[-0.5, 0.5]^2$ . The results obtained by solving SDP (15) with parameters  $d = 14$ ,  $\omega_h = 1.2$ , and  $\Delta h = 0.2$  are depicted in Fig 4.

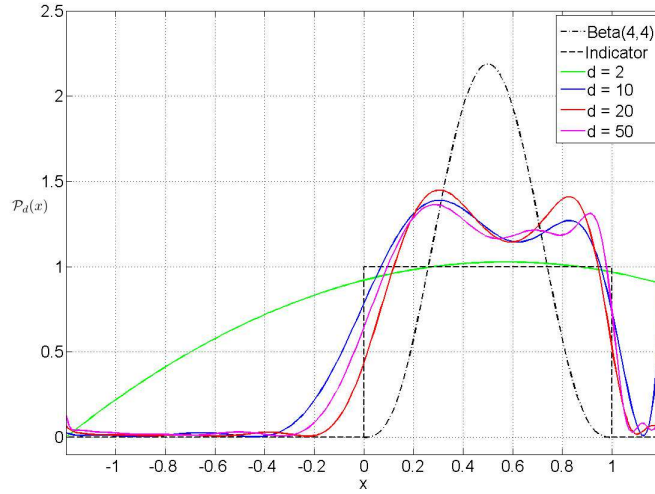


Fig. 3. Result of SDP in (15) For Example 2

#### IV. SUPPORT RECONSTRUCTION FOR UNIFORM MEASURES

In this section, we present a modification of our approach aimed specifically at uniform distributions. In the development to follow, we rely on a result in [13] which provides criteria under which polynomials vanish on the boundary of support of the uniform measure of interest. We now elaborate on this.

Define

$$\bar{M}_r(\mathbf{y})(i, j) = \frac{n + |i| + |j|}{n + |i|} y_{\alpha^{(i)} + \alpha^{(j)}}, \quad 1 \leq i, j \leq S_{n,r}, \quad (16)$$



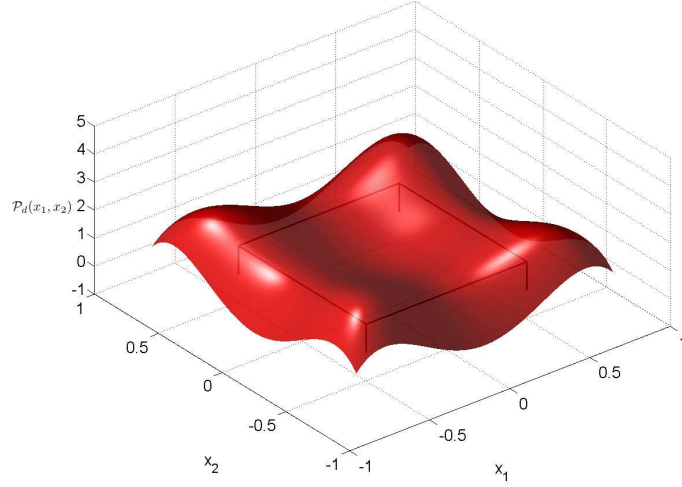


Fig. 4. Result of SDP in (15) For Example 3

where  $\mathbf{y} = \{y_\alpha\}_{\alpha \in \mathbb{N}^n}$  are the moments of the uniform distribution of interest. The results in [13] show that a polynomial  $\mathcal{P}(x)$  whose vector of coefficients  $\mathbf{p}$  is the eigenvector associated with zero eigenvalue of the matrix  $\bar{M}_r$ , vanishes on the boundary of support of measure. More precisely, under some technical conditions,

$$\bar{M}_r(\mathbf{y})\mathbf{p} = 0 \Rightarrow \mathcal{P}(x) = 0 \text{ for all } x \in \partial\mathcal{K} \quad (17)$$

where,  $\partial\mathcal{K}$  denotes the boundary of support set  $\mathcal{K}$ . However, without any additional constraints, this polynomial can also be zero in the interior of  $\mathcal{K}$  and, hence, it might not provide a good estimate of the support.

Nevertheless, one can take advantage of this property and modify our approach as follows.

*Problem 6:*

$$\mathbf{P}_6^* := \min_{p_\alpha, \sigma_j} \sum_{\alpha=0}^d p_\alpha y_{\mathcal{B}_\alpha} - \omega_h h + \omega_M \|\bar{M}_d(\mathbf{y})(\mathbf{p} - 1)\|_2 \quad (18)$$

$$\text{s.t. } \mathcal{P}_d(x) = \sum_{\|\alpha\|_1 \leq d} p_\alpha x^\alpha \quad (18a)$$

$$\mathcal{P}_d(x) = \sigma_0(x) + \sum_{j=1}^l \sigma_j(x) g_j(x) \quad (18b)$$

$$\sigma_j \in \Sigma^2[x]; j = 0, 1, \dots, l \quad (18c)$$

$$\deg(\sigma_0) \leq d; \quad (18d)$$

$$\deg(\sigma_j g_j) \leq d; j = 1, 2, \dots, l \quad (18e)$$

$$M_r((\mathcal{P}_d(x) - h)\mathbf{y}) \succcurlyeq 0 \quad (18f)$$

$$1 \leq h \leq 1 + \Delta h \quad (18g)$$

where,  $\omega_M$ ,  $\omega_h$  and  $\Delta h$  are positive design parameters,  $\mathbf{p} = \{p_\alpha\}_{\alpha \in \mathbb{N}^n}$  denotes the vector of polynomial coefficients and  $\|\cdot\|_2$  denotes the  $l_2$  norm.

In fact in (18), we aim at “pushing” the coefficients of the polynomial  $(\mathcal{P}(x) - 1)$  as close as possible to the null space of  $\bar{M}_d$  by minimizing the term  $\|\bar{M}_d(\mathbf{p} - 1)\|_2$ . In this case obtained polynomial  $\mathcal{P}_d(x)$



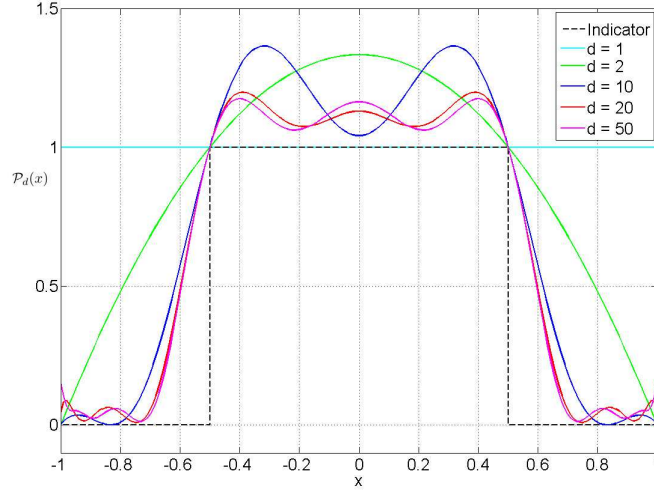


Fig. 5. Result of SDP in (18) For Example 1

becomes close to one at the boundary of support while we still aim at having  $\mathcal{P}_d$  larger than one inside the support.

To show the effectiveness of proposed method, we reconstruct the support for the measure of Example 1 by solving the SDP (18) with parameter  $\omega_M = 10$ . The obtained result are depicted in Fig 5, where semialgebraic set  $\mathcal{K}_d = \{x \in \mathbb{R} : \mathcal{P}(x)_d \geq 1\}$  for any polynomial order  $d \geq 2$  exactly reconstructs the support of measure.

**Example 4:** To further show the effectiveness of our approach, we now consider a uniform distribution with disconnected support. More precisely, we aim at estimating the support of a uniform probability measure over the union of the sets  $[-0.8, -0.4]$  and  $[0.3, 0.7]$ . We assume that  $\mathcal{B} = [-1, 1]$  and use moments up to order  $2d$ . The results obtained by solving SDP (18) with parameters  $\omega_h = 1.2$ ,  $\omega_M = 10$  and  $\Delta h = 0.2$  are depicted in Fig 6, where one can see that the semialgebraic set  $\mathcal{K}_d = \{x \in \mathbb{R} : \mathcal{P}_d(x) \geq 1\}$  for  $d \geq 4$  exactly reconstructs the support of measure.

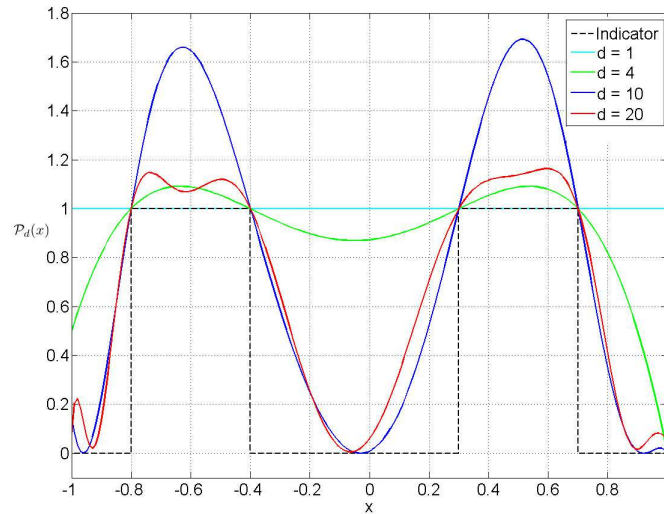


Fig. 6. Result of SDP in (18) For Example 4

## V. CONCLUSION

In this paper, we present a novel approach to the problem of reconstruction of support of measures from their moments. A sequence of semidefinite relaxations is provided whose solution converge to the support of the measure of interest. Examples are provided that show that one does obtain a good approximation of support using only a finite number of moments. Further research effort is now being put on developing methods for support reconstruction for specific classes of measures which have provable performance.

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